Space-Time Localized Radial Basis Function Collocation Methods for PDEs

Alfa Heryudono

Department of Mathematics
University of Massachusetts Dartmouth

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Dealing with Time-Dependent PDEs for RBF Methods

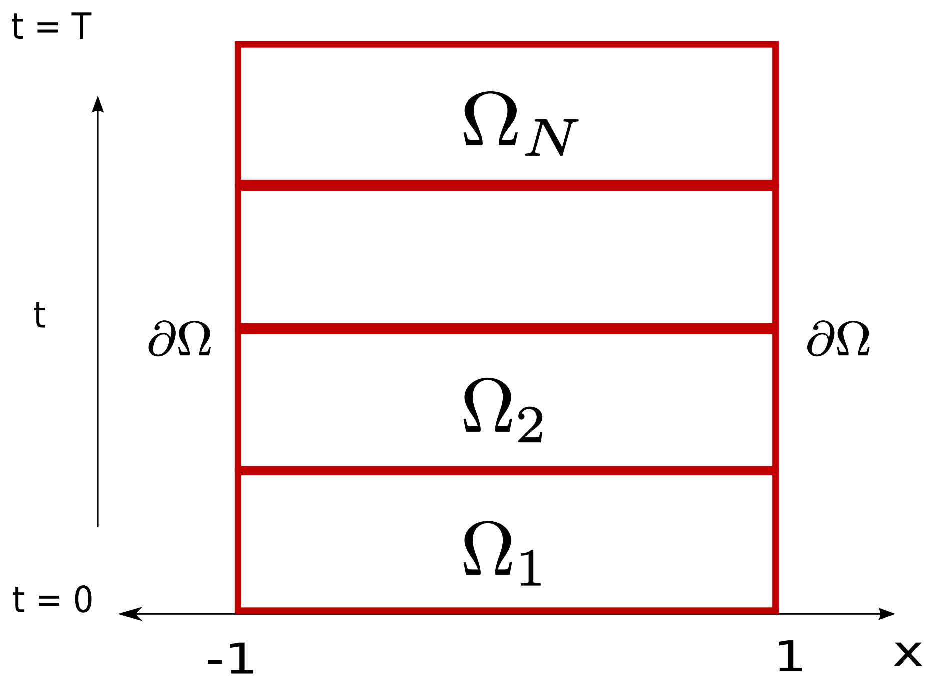
Method of Lines

- RBF discretization in space + common ODE solver in time.
- Min changes of PS/FD codes: replace differentiation matrices with RBF versions (Global, RBF-FD, RBF-PU, etc).
- PS/FD treatments for BCs: Strip-rows, Strip-rows move over columns, fictitious pts/ rect projection (for multiple bcs), penalty, etc.
- Stability for linear pde case: Eigenvalue and Pseudospectra.

Simultaneous Space-Time RBF

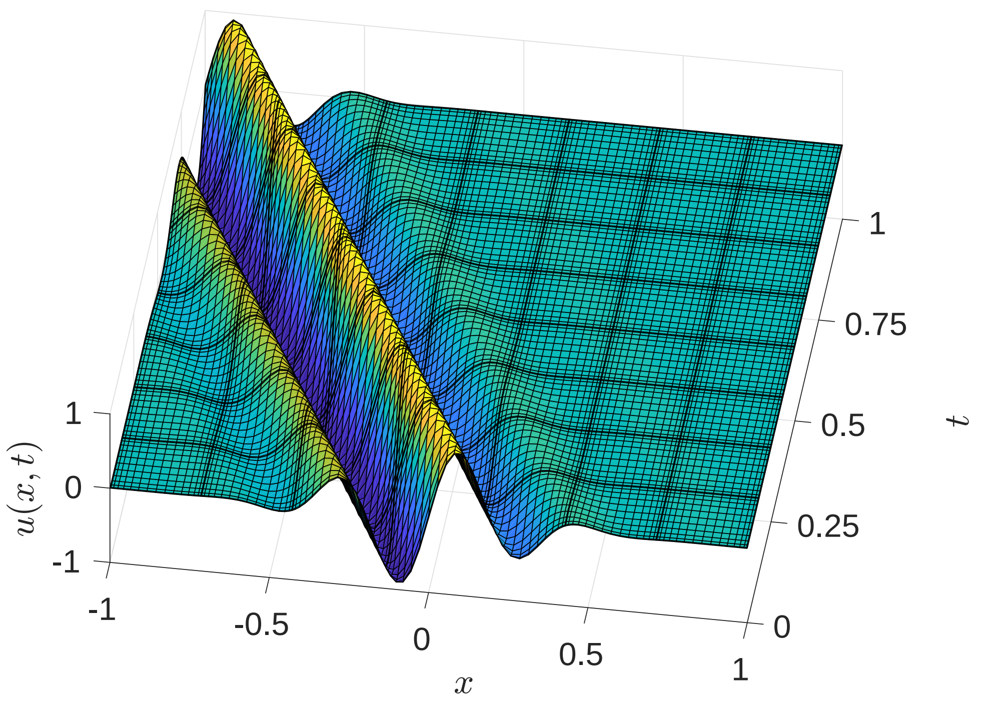
- Boundary value collocation problem in space-time domain. Time is treated as another space variable. RBF-BVP solver have been studied for quite a while.
- Less worry about choosing ODE solver based on PDE types.
- Adaptivity, moving boundary, and BCs: same treatments as in BVP cases.
- No need to rewrite the pde due to var trans (e.g in moving boundary case).
- Analyzing stability is not clear (e.g. in moving boundary case).
- Might be expensive to solve (e.g. finding preconditioner, non-linear case).

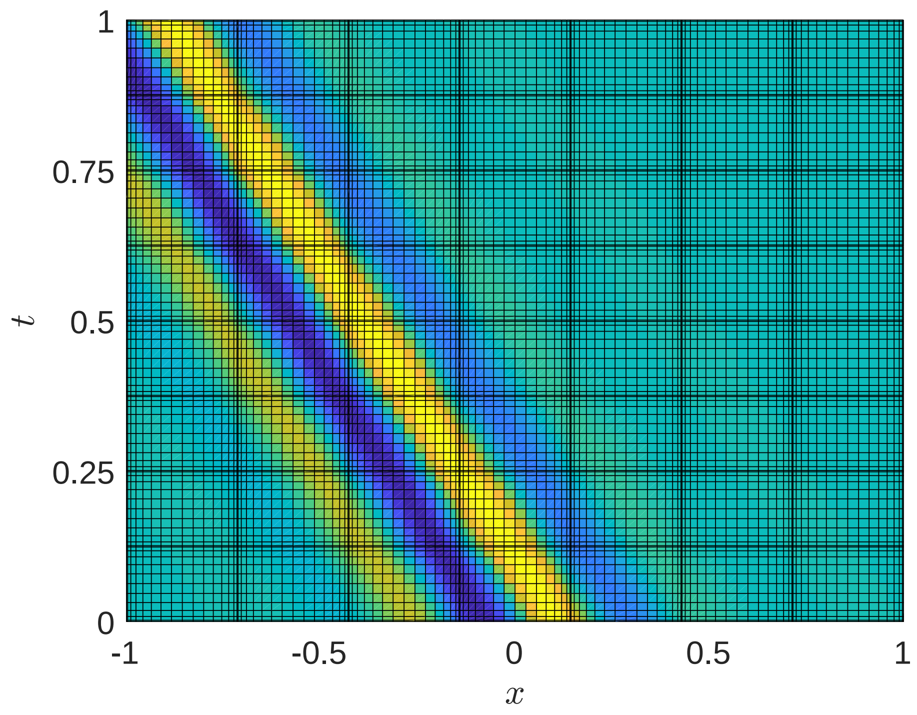
Space-Time PS Collocation Method: 1D+t linear case



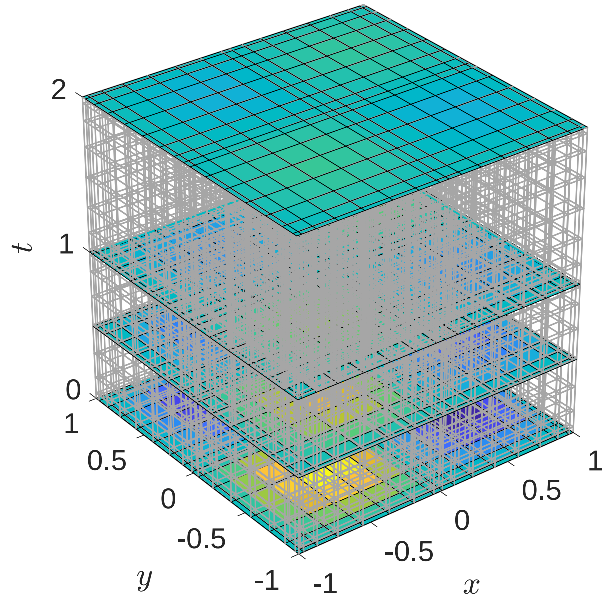
$$\begin{aligned} PDE: &\quad u_t = u_x \\ &\quad (x,t) \in [-1,1) \times (0,T] \\ IC: &\quad u(x,0) = f(x) \\ BC: &\quad u(1,t) = g(t) \end{aligned}$$

Use PS or Block PS (Driscoll-Fornberg) to create differentiation matrices.



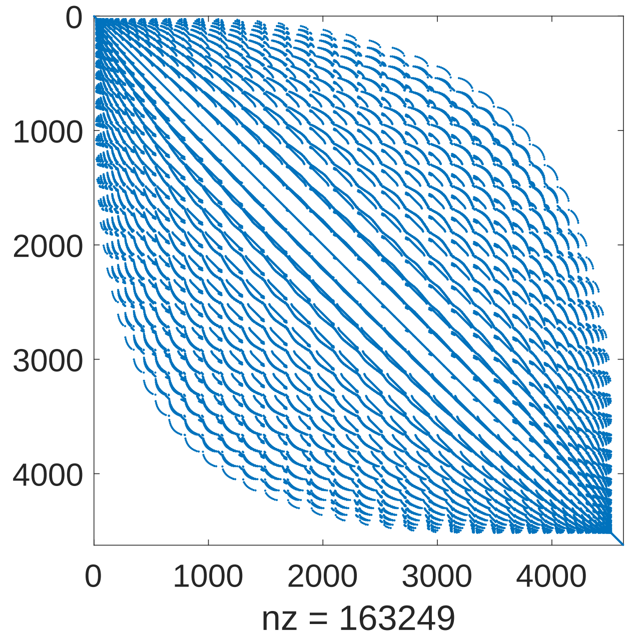


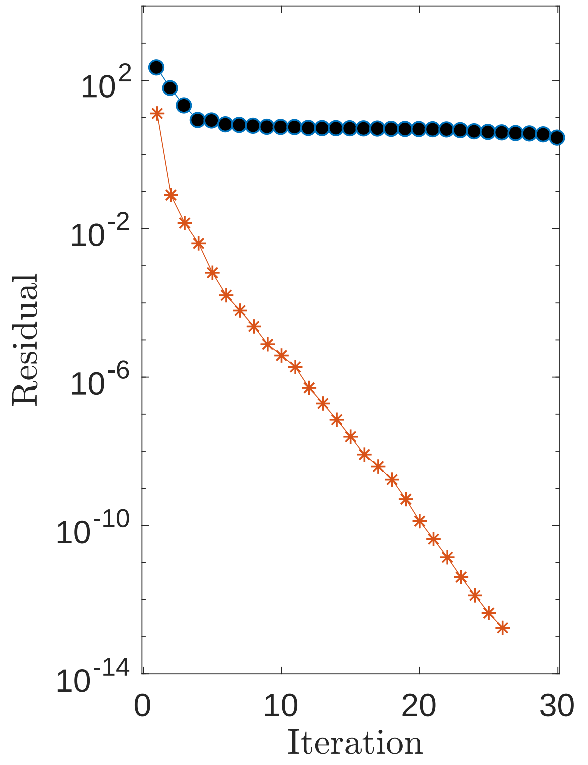
Space-Time PS Collocation Method: 2D+t, linear case



$$
\begin{aligned}
PDE: &\quad u_t = \Delta u + F(x,y,t)\\
&\quad (x,y,t) \in \Omega \times (0,T]\\
IC: &\quad u(x,y,0) = f(x,y)\\
BC: &\quad u(\partial\Omega,t) = g(\partial\Omega,t)
\end{aligned}
$$

kron's disease is worse in $2D + t$ case.





```
P = symrcm(PLinop);

L = gpuArray(Linop(P,P));

PL = gpuArray(PLinop(P,P));

r = gpuArray(rhs(P));

MAXITER = 30; TOL = 1e-14; RESTART = [];

[Ugpu,FLAG,RELRES,ITER,RESVEC] = ...

gmres(L,r,RESTART,TOL,MAXITER,PL);

U(P) = gather(Ugpu);
```

Space-Time PS Collocation Method: 1D+t, nonlinear case

Human tear film dynamics: 1D model: see H. et. al 2007

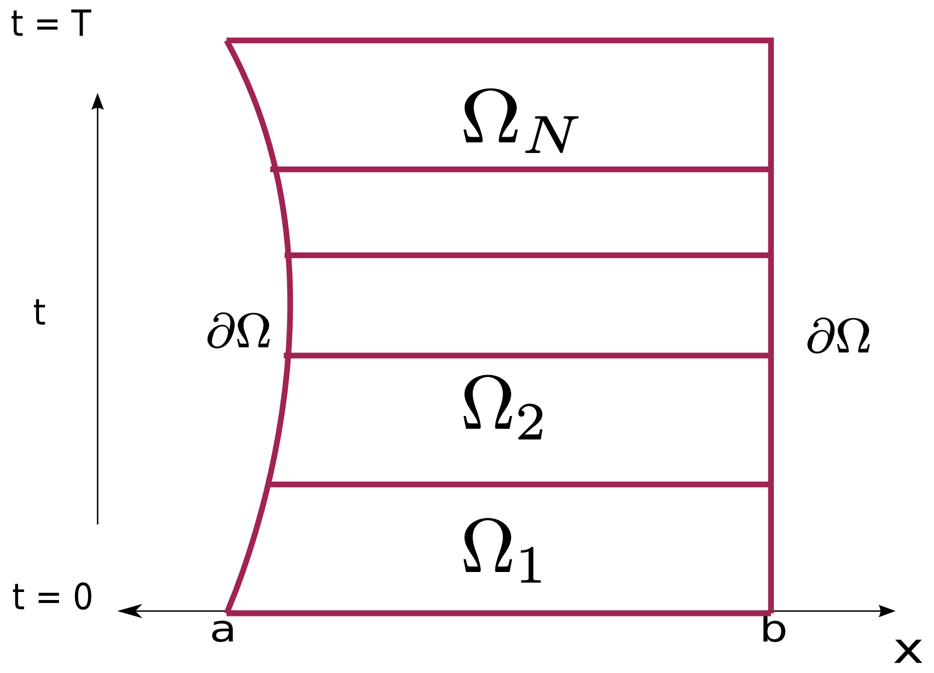
$$h_t + q_x = 0 \text{ on } X(t) \le x \le 1,$$

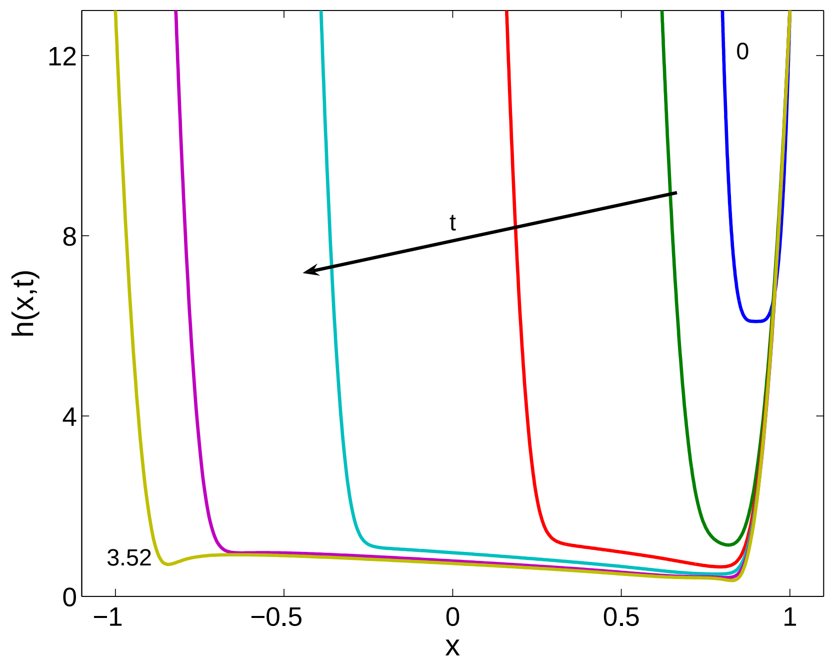
where

$$q(x,t) = Sh_{xxx}\left(\frac{h^3}{3} + \beta h^2\right)$$

Boundary conditions

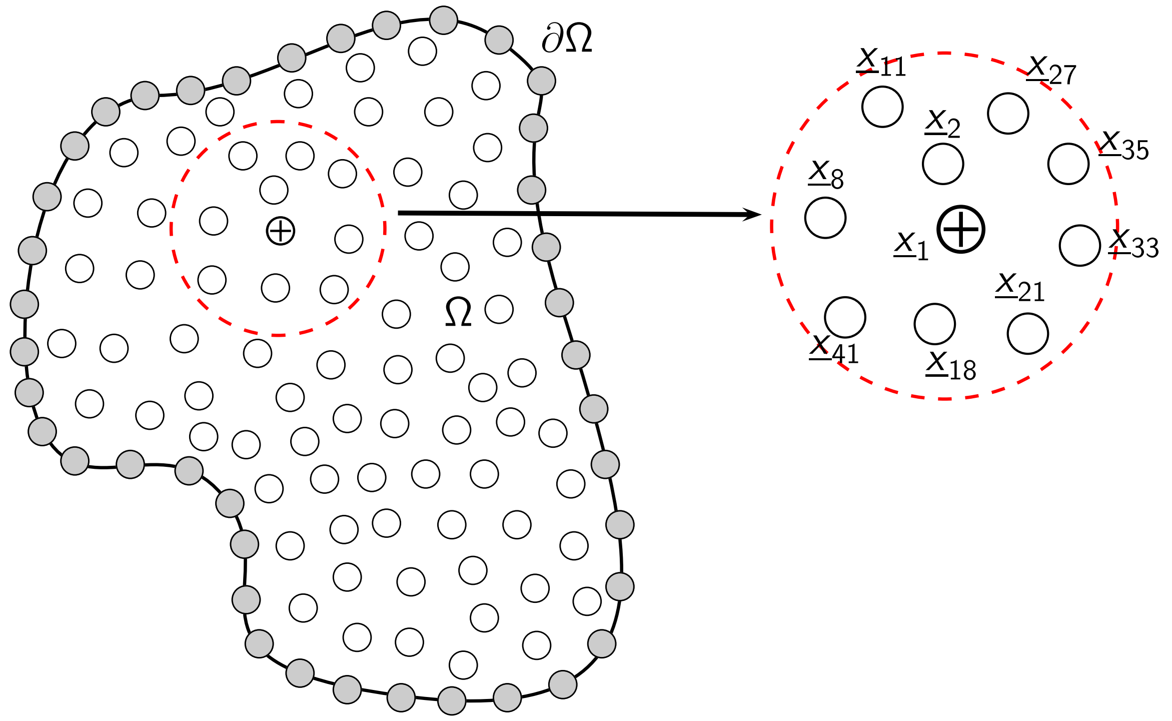
$$h(X(t),t) = h(1,t) = h_0 \quad q(X(t),t) = X_t h_0 + Q_{top} \quad q(1,t) = -Q_{bot}.$$





Advance the solution in space-time domain: Slab by Slab (Show MATLAB).

RBF-FD Differentiation Matrices



$$s_j(\underline{x}) = \sum_{k=1}^{n_{\text{loc}}} \lambda_k \phi^k(\underline{x}),$$

where $\phi^k(\underline{x})$ is a radial basis function centered at $\underline{x}_k$.
Or in Lagrange formulation as

$$s_j(\underline{x}) = \sum_{k=1}^{n_{\text{loc}}} \Psi^k(\underline{x}) u_k,$$

where

$$\underline{\Psi} = \begin{bmatrix}\Psi^1(\underline{x}) & \cdots & \Psi^{n_{\text{loc}}}(\underline{x})\end{bmatrix} = \begin{bmatrix}\phi^1(\underline{x}) & \cdots & \phi^{n_{\text{loc}}}(\underline{x})\end{bmatrix}\begin{bmatrix}A^{-1}\end{bmatrix},$$

$$\underline{\Psi}_x = \begin{bmatrix}\Psi^1_x(\underline{x}) & \cdots & \Psi^{n_{\text{loc}}}_x(\underline{x})\end{bmatrix} = \begin{bmatrix}\phi^1_x(\underline{x}) & \cdots & \phi^{n_{\text{loc}}}_x(\underline{x})\end{bmatrix}\begin{bmatrix}\mathrm{A}^{-1}\end{bmatrix},$$

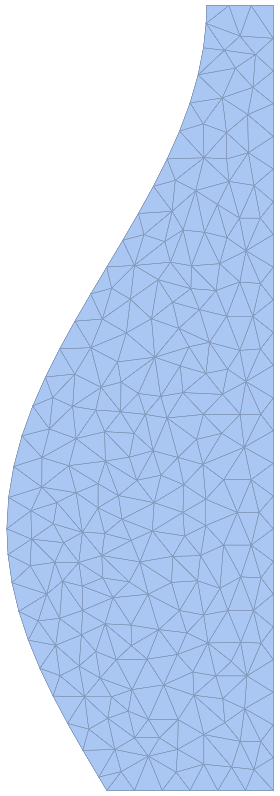
The matrix A with entries

$$a_{\ell k} = \phi^k(\underline{x}_\ell), \qquad \ell, k = 1, \ldots, n_{\text{loc}}$$

is *local RBF interpolation matrix*.
BYODM: Bring Your Own Differentiation Matrices

Getting the space-time domain

This is probably for programming on a lazy Sunday: Use MATHEMATICA's **DiscretizeRegion** family commands. Surprisingly, MATHEMATICA has many built-in funky domains too. This is also useful if you want to compare results with finite-element.



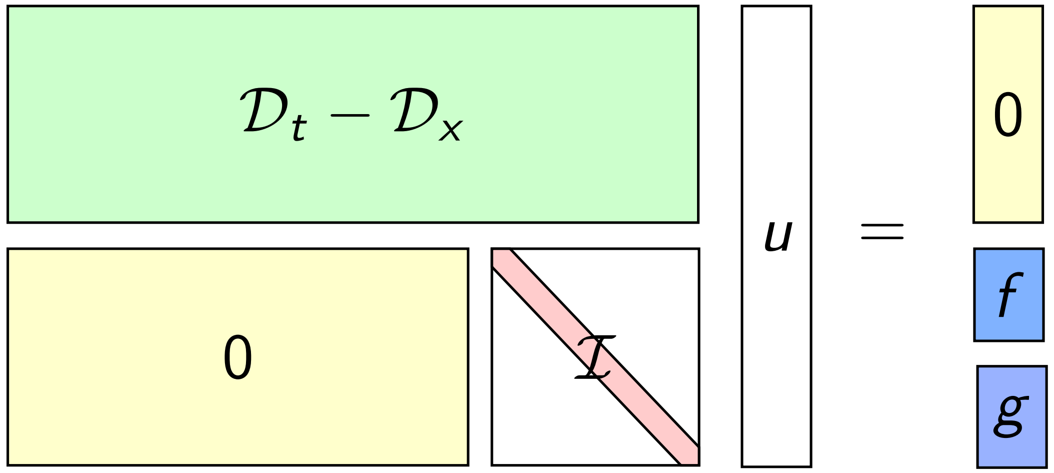
```
R = ImplicitRegion[-0.6 Sin[t] <= x, {{x, -1, 1},
       {t, 0, 1.5 Pi}}];
ev = DiscretizeRegion[R];
pts = MeshCoordinates[ev];
Export["spacetimedom.mat", pts];
```

To obtained boundary points, you can use MATHEMATICA or **boundary** command in MATLAB.

t+1D Advection Example

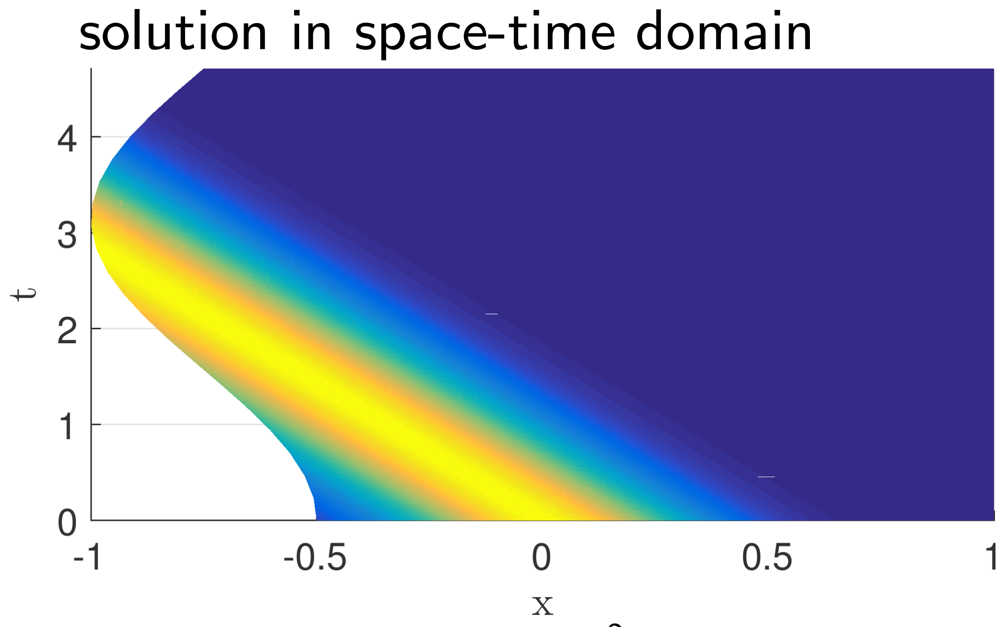
$$
\begin{aligned}
PDE: \quad & u_t = u_x \\
& (x,t) \in [X(t),1) \times (0,T] \\
IC: \quad & u(x,0) = f(x) \\
BC: \quad & u(1,t) = g(t)
\end{aligned}
$$

IMQ-RBF: $\frac{1}{\sqrt{1+(\varepsilon r)^2}}$. $r^2 = (x-x_i)^2 + (t-t_i)^2$

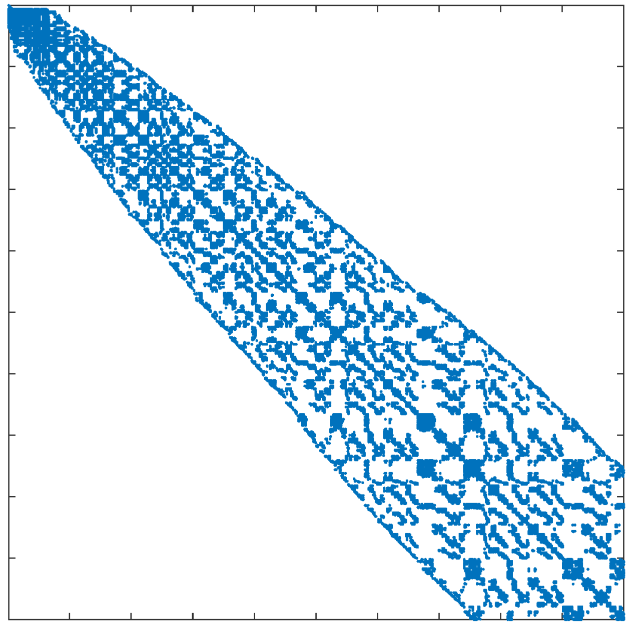


```
P = symrcm(L); u(P) = L(P,P)\RHS(P);
```
or

```
MAXITER = 20; TOL = 1e-13; RESTART = [];
[ML,MU] = ilu(L(P,P),struct('type','ilutp','droptol',1e-6));
u(P) = gmres(L(P,P),RHS(P),RESTART,TOL,MAXITER,ML,MU);
```



$f(x) = e^{-10(x-0.15+0.35y)^2}$, $g(t) = 0$



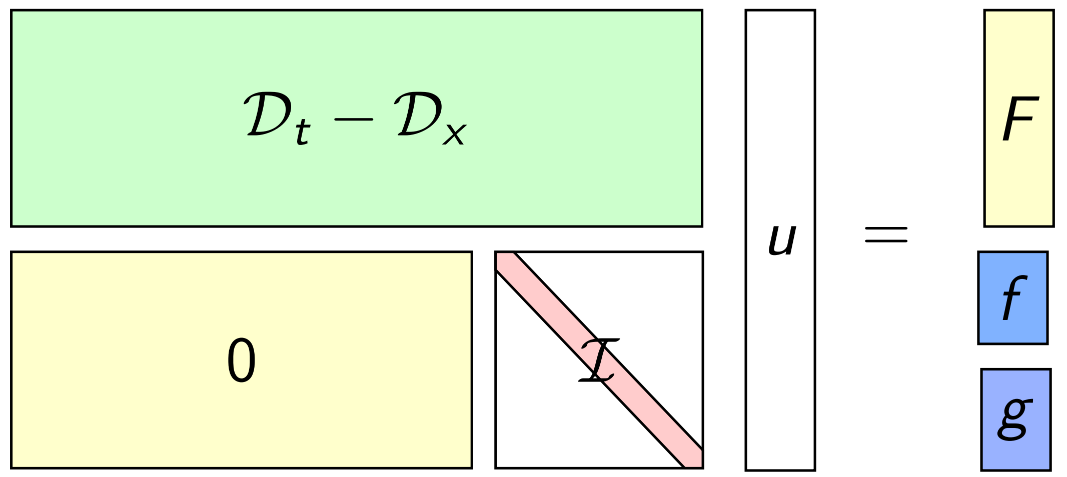
portion of system matrix
after applying MATLAB symrcm

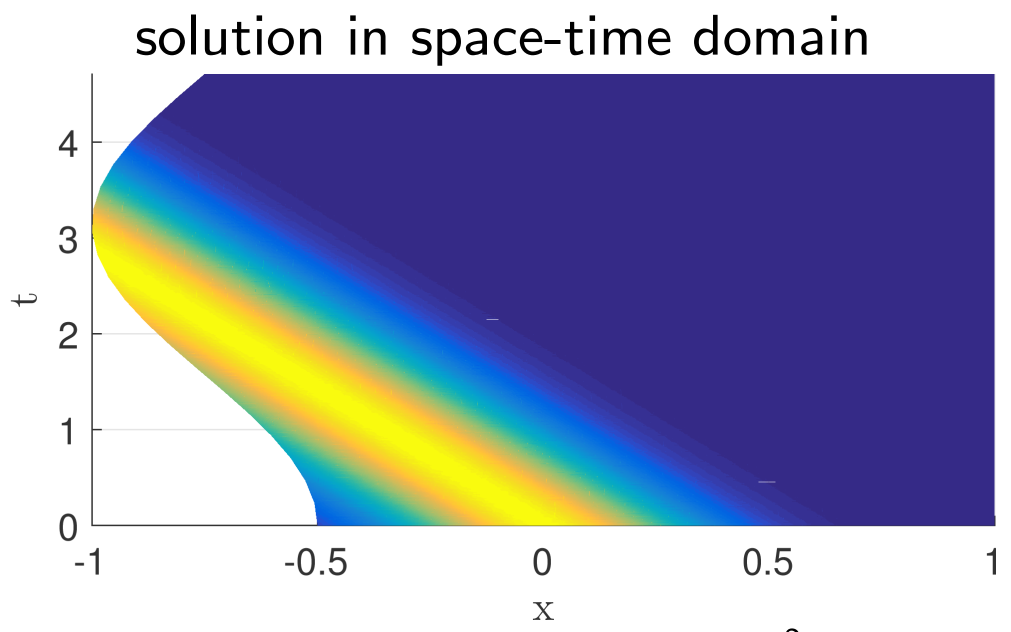
t+1D Advection Example

$$\begin{aligned} PDE: &\quad u_t = u_x + F(x,t) \\ &\quad (x,t) \in [X(t),1) \times (0,T] \\ IC: &\quad u(x,0) = f(x) \\ BC: &\quad u(1,t) = g(t) \end{aligned}$$

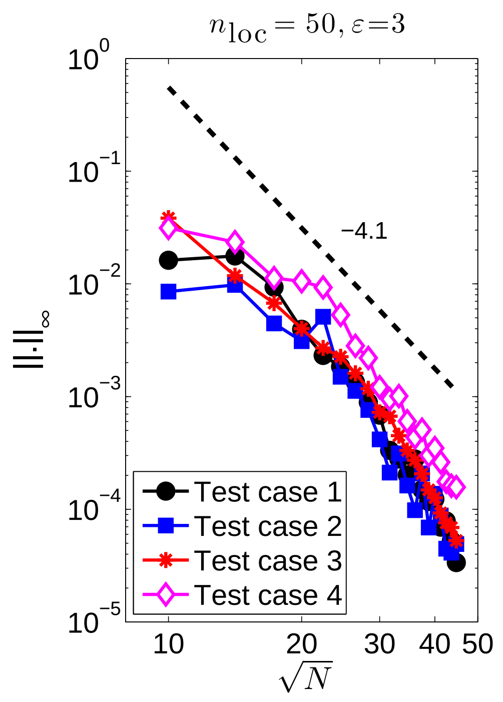
IMQ-RBF: $\frac{1}{\sqrt{1+(\varepsilon r)^2}}$. $r^2 = (x-x_i)^2 + (t-t_i)^2$

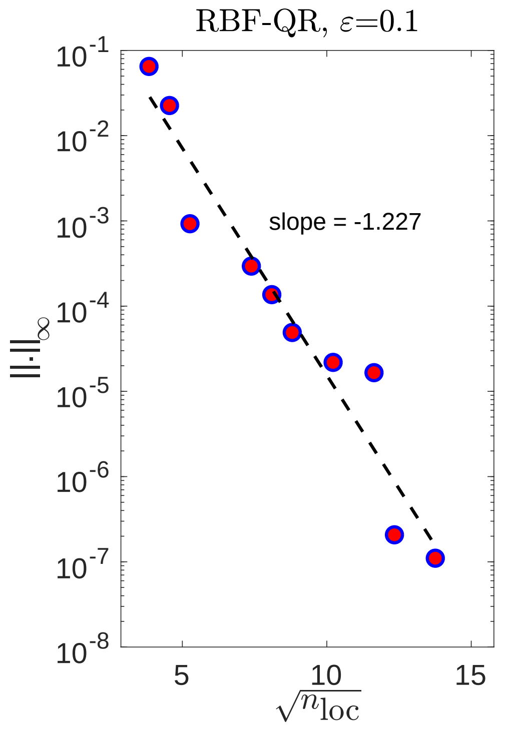
Get RBF-QR diffmat from Elisabeth's website.





$$f(x) = e^{-10(x-0.15+0.35y)^2}$$

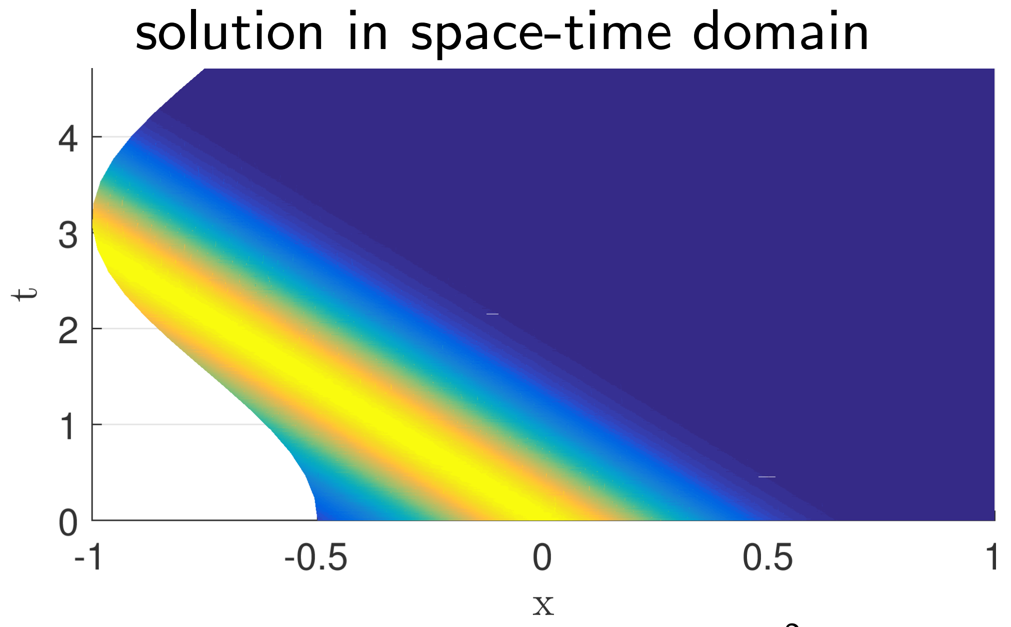




t+1D Advection with Variable Speed Example

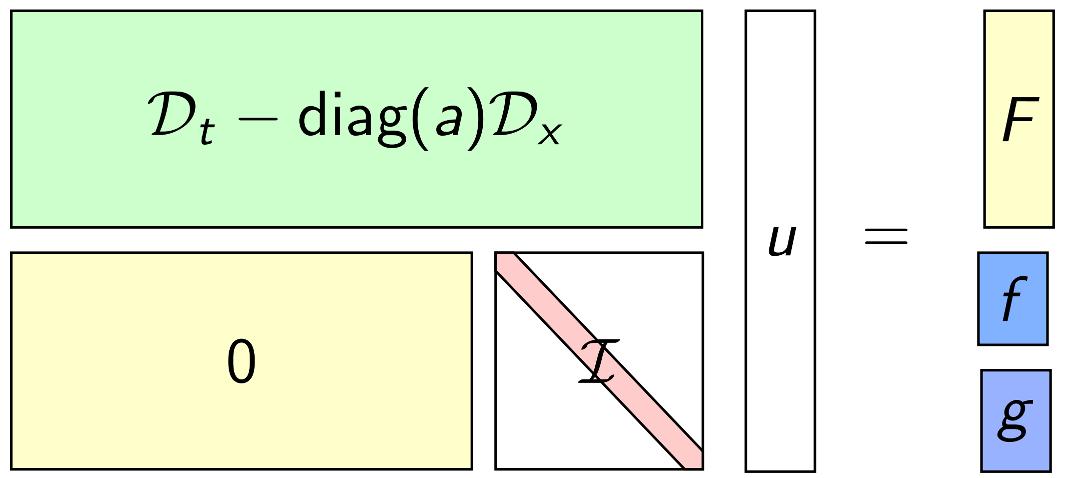
$$
\begin{aligned}
PDE: &\quad u_t = a(x,t)u_x + F(x,t)\\
&\quad (x,t) \in [X(t),1) \times (0,T]\\
IC: &\quad u(x,0) = f(x)\\
BC: &\quad u(1,t) = g(t)
\end{aligned}
$$

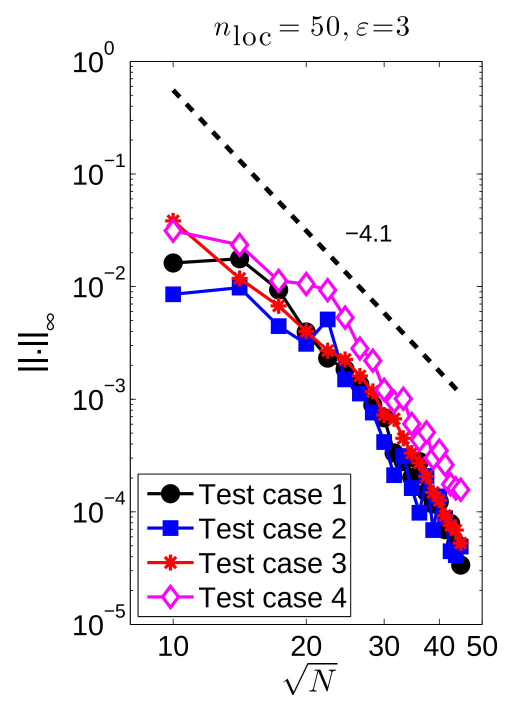
$$a(x,t) = \exp((1+t)(1+\cos(3x))$$

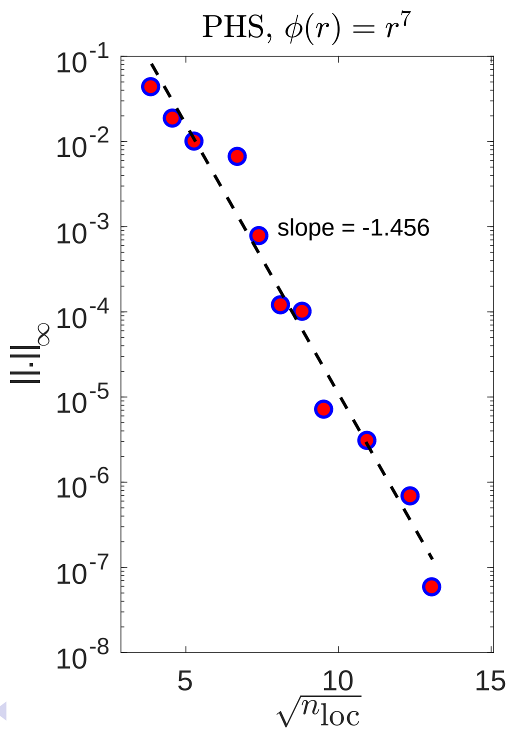


$$f(x) = e^{-10(x-0.15+0.35y)^2}$$

Bribe Varun for PHS diffmat.

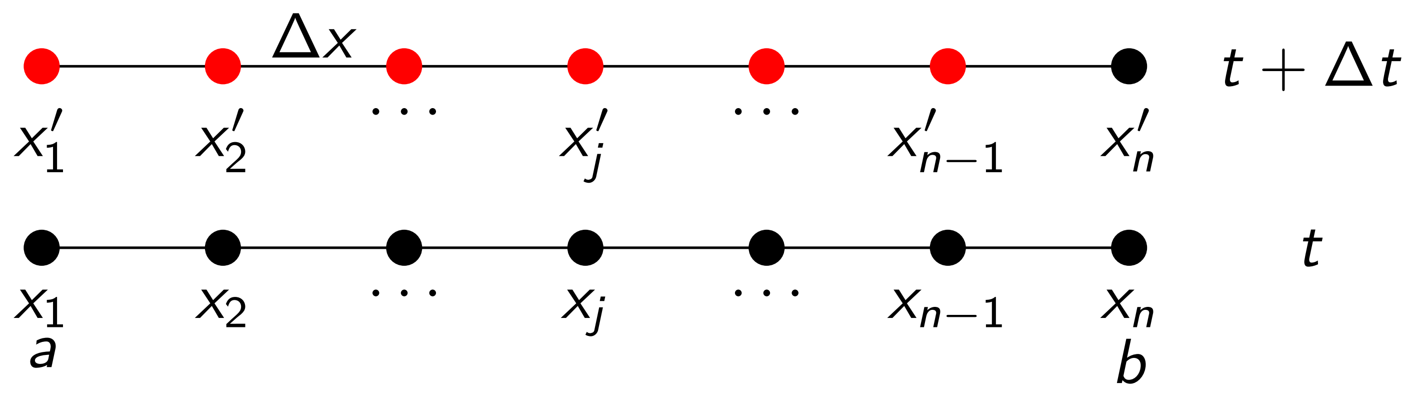






Analyzing Stability ?

Let's take a look at one step (2 levels) space-time global RBF method.

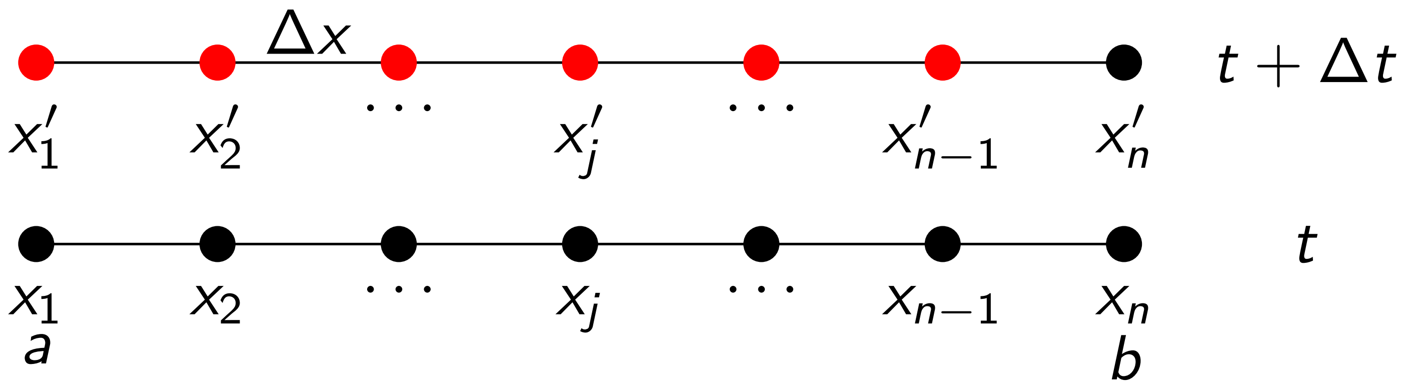


for a simple 1-D advection equation

$$\begin{aligned}\text{PDE:} \quad & \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \quad \text{for } x \in [a, b) \\ \text{IC:} \quad & u(x,0) = u_0(x) \quad \text{when } t = 0 \\ \text{BC:} \quad & u(b,t) = g(t) \quad \text{at } x = b\end{aligned}$$

$$u(x) = \sum_{j=1}^{n} \lambda_j \phi(\varepsilon \|x - x_j\|) + \sum_{j=1}^{n} \lambda_j' \phi(\varepsilon \|x - x_j'\|),$$

where $\{x_j\}$ and $\{x_j\}$ are centers at the old time level and new time level respectively.



Our goal is to find the unknowns $\{\lambda_j\}$ and $\{\lambda_j'\}$. This can be done by enforcing initial and boundary data and satisfying the PDE at the interior points that lead to solving system of linear equations

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \\ \hline \lambda_1' \\ \vdots \\ \lambda_n' \end{array}\right] = \left[\begin{array}{c} u_o(x_1) \\ \vdots \\ u_o(x_n) \\ \hline 0 \\ \vdots \\ g(t) \end{array}\right]$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \\ \hline \lambda'_1 \\ \vdots \\ \lambda'_n \end{array}\right] = \left[\begin{array}{c} u_o(x_1) \\ \vdots \\ u_o(x_n) \\ \hline 0 \\ \vdots \\ g(t) \end{array}\right]$$

The block matrices A, B, C, D are all $n \times n$ matrices with elements:

- $A_{ij} = \phi(\varepsilon\|x_i - x_j\|)$
- $B_{ij} = \phi(\varepsilon\|x_i - x'_j\|)$
- $C_{ij} = \mathcal{L}\phi(\varepsilon\|x'_i - x_j\|)$
- $D_{ij} = \mathcal{L}\phi(\varepsilon\|x'_i - x'_j\|)$

for all $i, j = 1, \cdots, n$ and $\mathcal{L} := \frac{\partial}{\partial t} - \frac{\partial}{\partial x}$. The last row C and D must be slightly modified to satisfy the boundary condition at $x'_n = b$.

Amplification Matrix and Stability Region

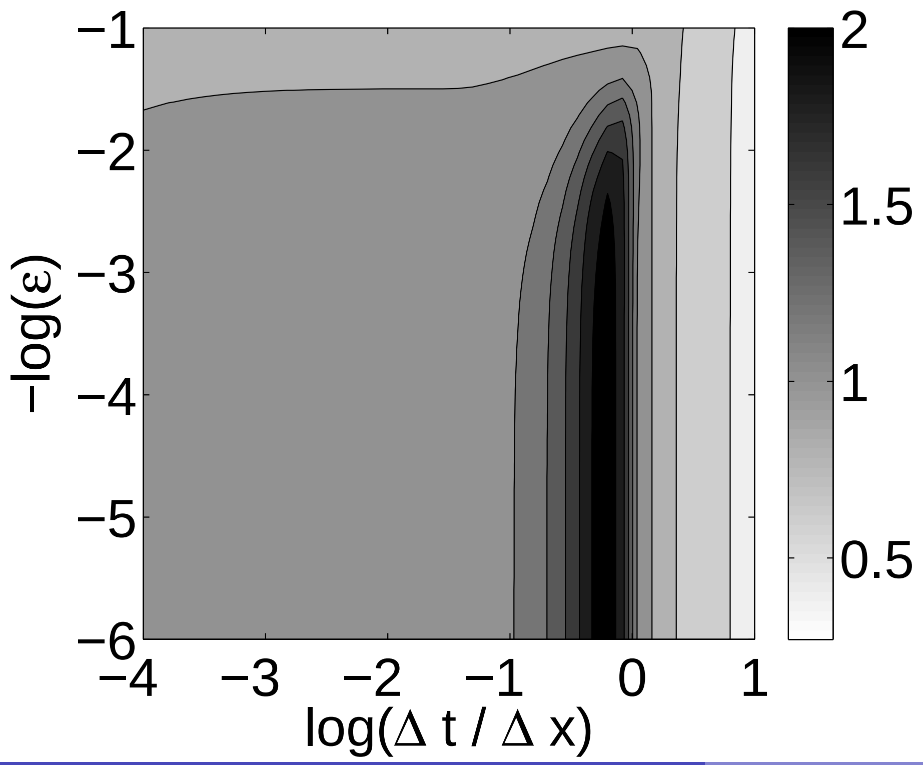
The process of marching in time to the new time level is given by

$$\begin{bmatrix} u(x_1') \\ \vdots \\ u(x_n') \end{bmatrix} = \begin{bmatrix} & & \\ & G & \\ & & \end{bmatrix} \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_n) \end{bmatrix}$$

where

$$G = \left[\; B \mid A \;\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} \left[\begin{array}{c} I \\ \hline 0 \end{array}\right],$$

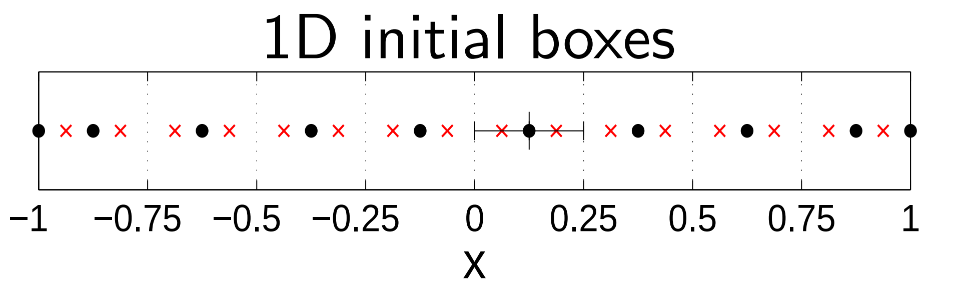
and I is an $n \times n$ identity matrix. The method is numerically stable if spectral radius $\rho(G) < 1$.

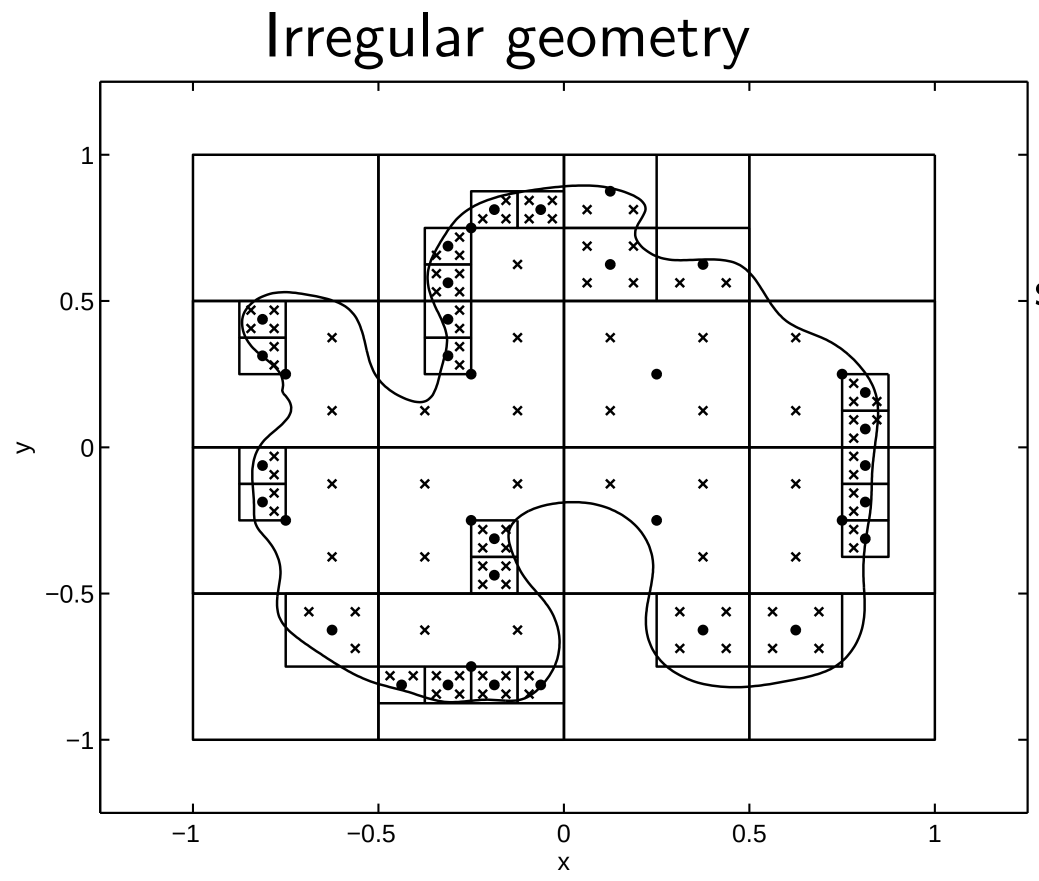


IMQ, $g(t) = 0$, $N = 50$: to avoid blowing up the solution, the ratio of $\Delta t/\Delta x$ vs shape parameter ε must be away from the darker alley in the the stability region, i.e we want to avoid $\rho(G) \geq 1$

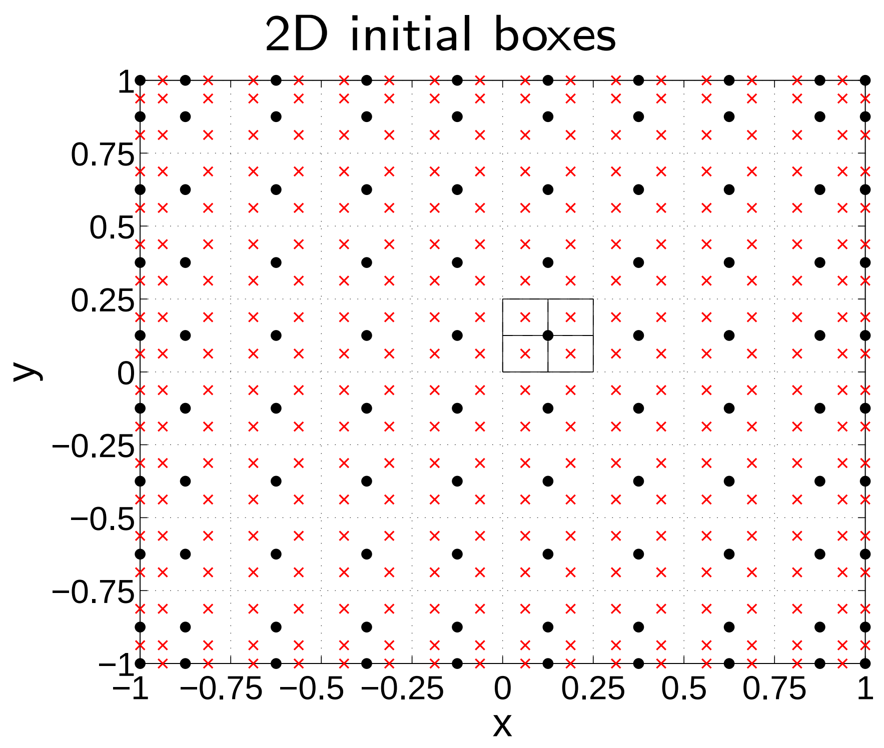
Adaptivity for BVP based on Residual subsampling

1. Initial coarse collection of nonoverlapping regular boxes in R^d that cover the domain Ω of interest.
2. Geometric adaptation.
3. Refining and Coarsening steps.



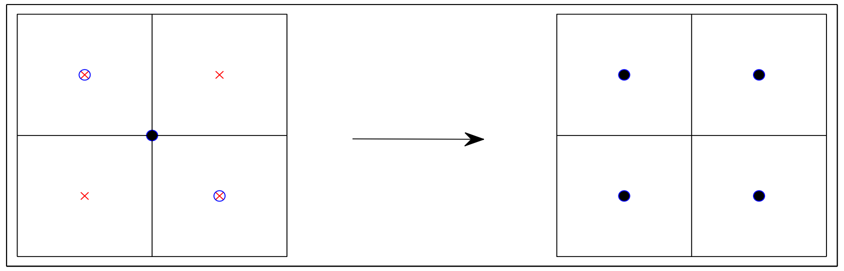


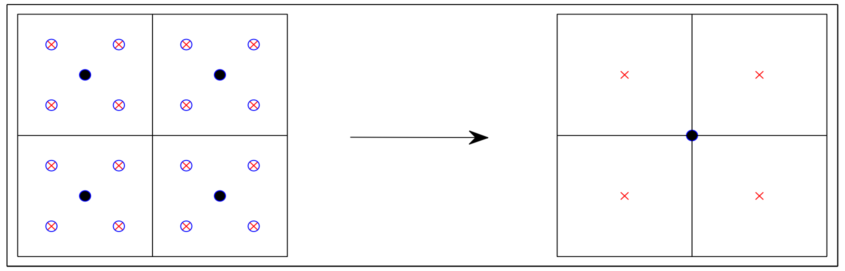
see



Driscoll & H (2007)

Rules of refining and coarsening centers

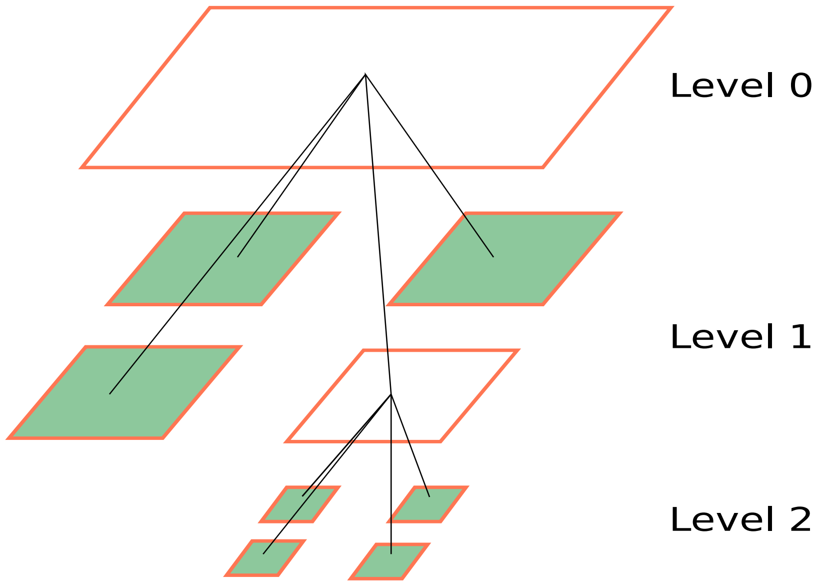




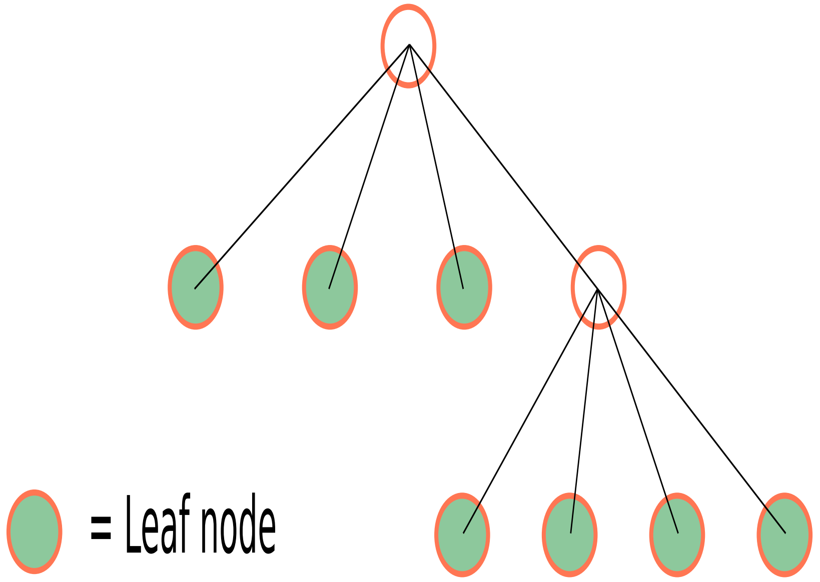
Refinement strategy: converting all check points if any of them have residual errors are greater than θ_r described as $\otimes$ into RBF centers as dots and remove its parent.

Coarsening strategy: reactivate all RBF centers if all of its grand children have residual errors less than θ_c described as $\otimes$.

a layout of quadtree.

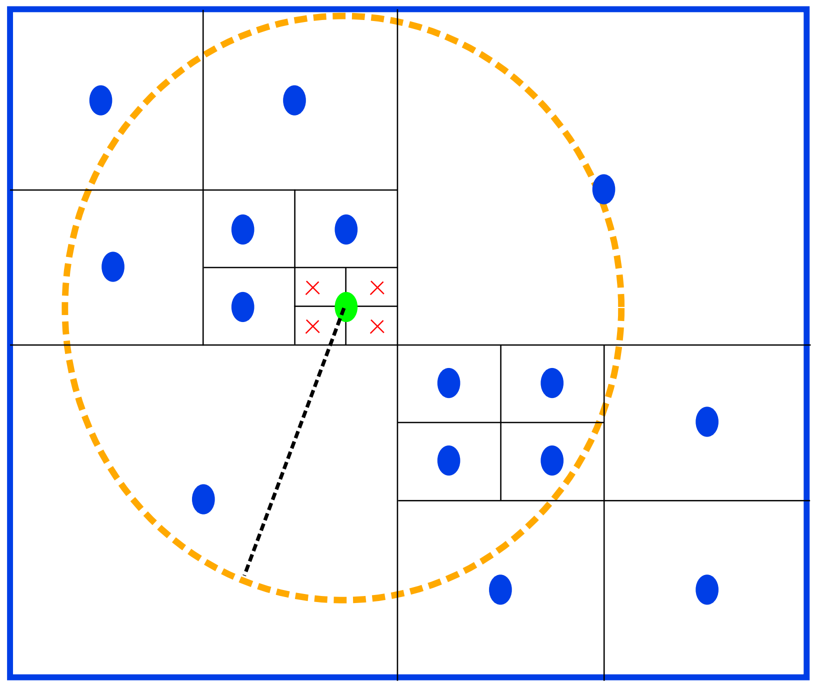


With this rule, centers are located as leaves.



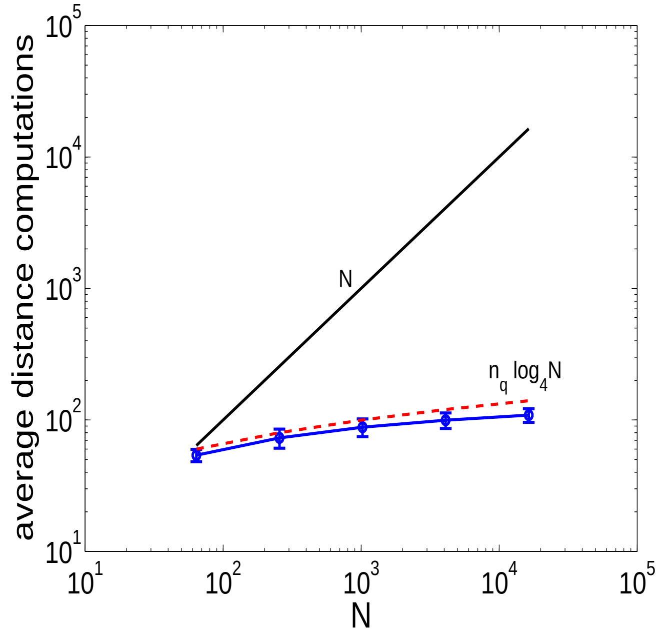
Depth first search algorithm

- Pruning device to save computing pairwise distances.:$\mathcal{O}(n_q \log(N))$ instead of $\mathcal{O}(N)$ per query point.
- Partial updates for lists of neighbors.
- Embarassingly parallel neighbors' search.

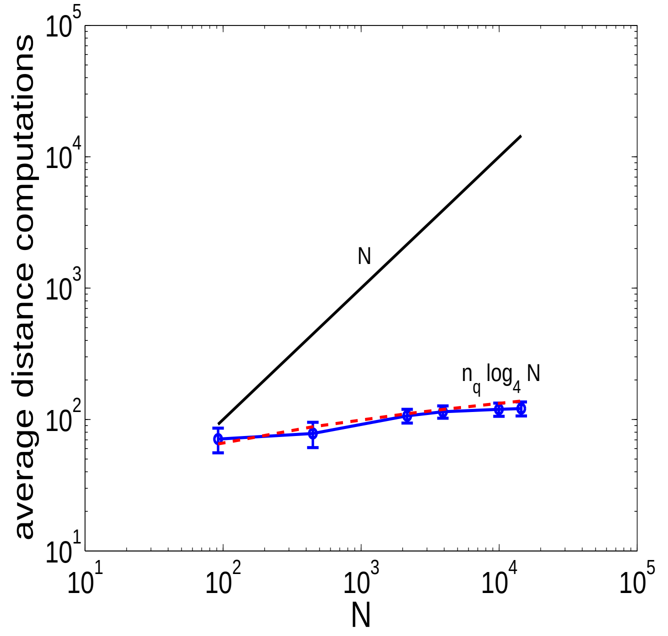


Values at $\times$ are computed using local RBF interpolant of the box whose midpoint is the parent node of the check points.

Uniform nodes distribution



Some non-uniform nodes distribution

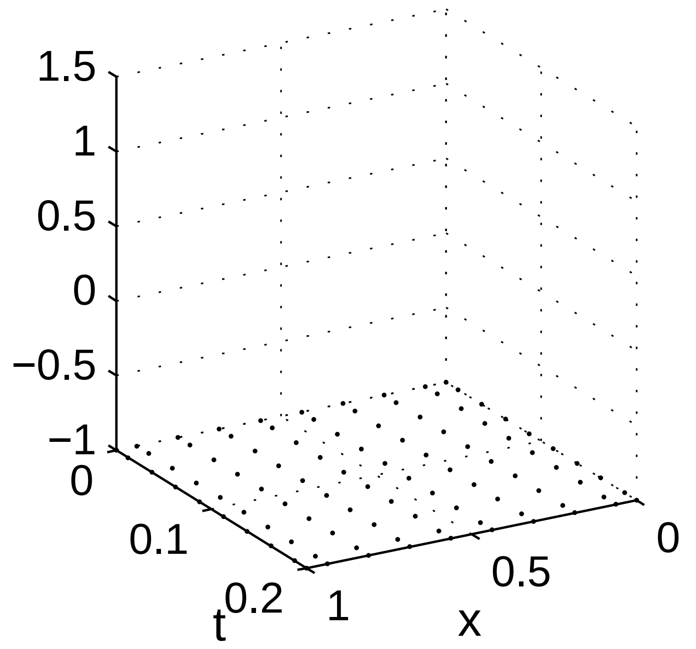


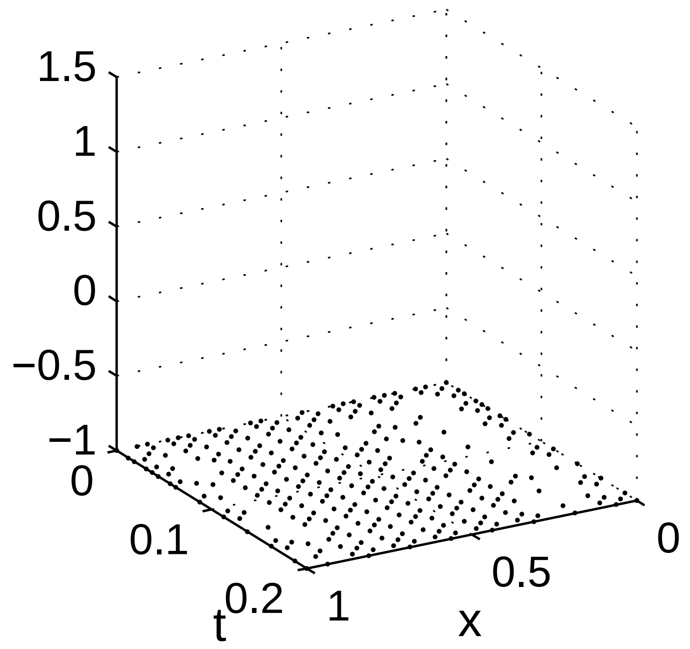
t+1D Nonlinear Example

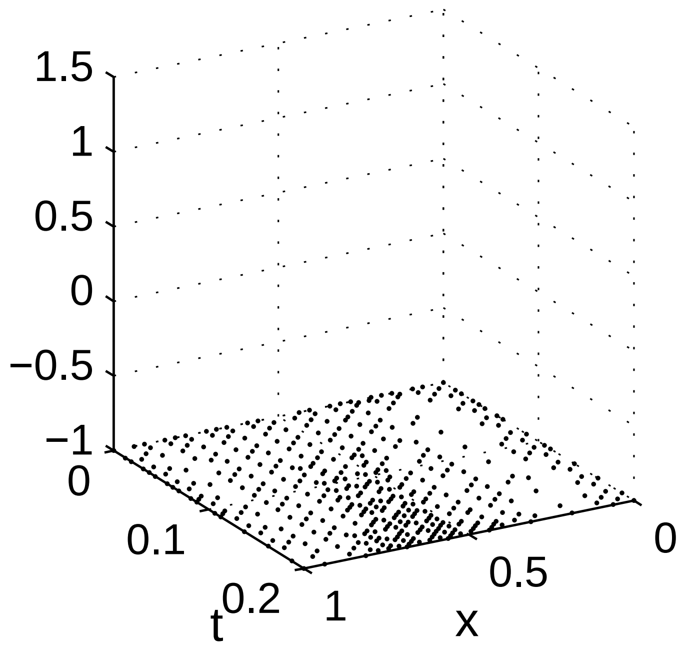
Burgers' Equation

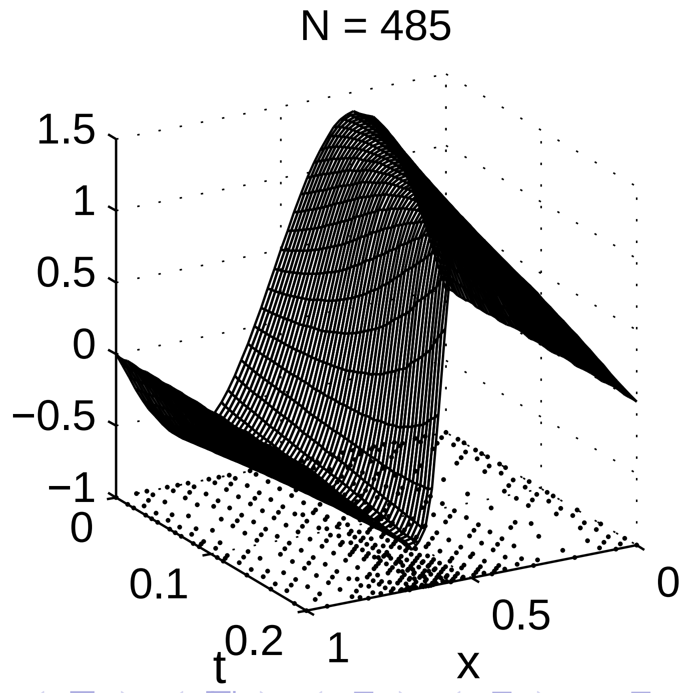
$$\upsilon u_{xx} - u u_x = u_t, \qquad 0 < x < 1$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = \sin(2\pi x) + \tfrac{1}{2}\sin(\pi x).$$
$$\text{where, } \upsilon = 10^{-3}$$

MATLAB's **fsolve** is used to solve the nonlinear system. Jacobian file is provided.









Dealing with Multiple Boundary Conditions

PDE : Tear film PDE in terms of h

$(x,t) \in [X(t),1) \times (0,T]$

IC : $h(x,0) = f(x)$

BC : $h(1,t) = h(X(t),t) = h_0$

$h_{xxx}(1,t) = g_1(t)$

$h_{xxx}(X(t),t) = g_2(X(t),t)$

PDE : $h_t = Sq_x$,S is a constant

$q =$ nonlinear flux

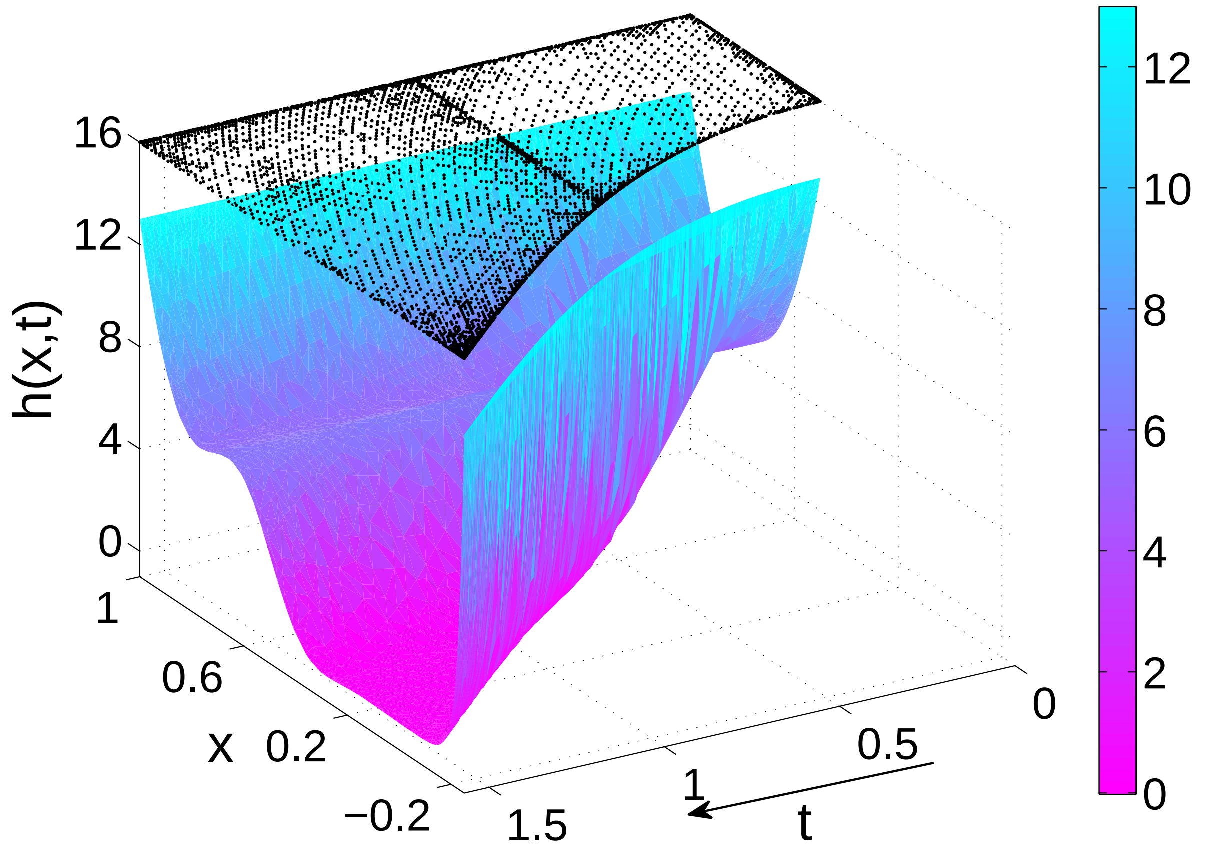
$(x,t) \in [X(t),1) \times (0,T]$

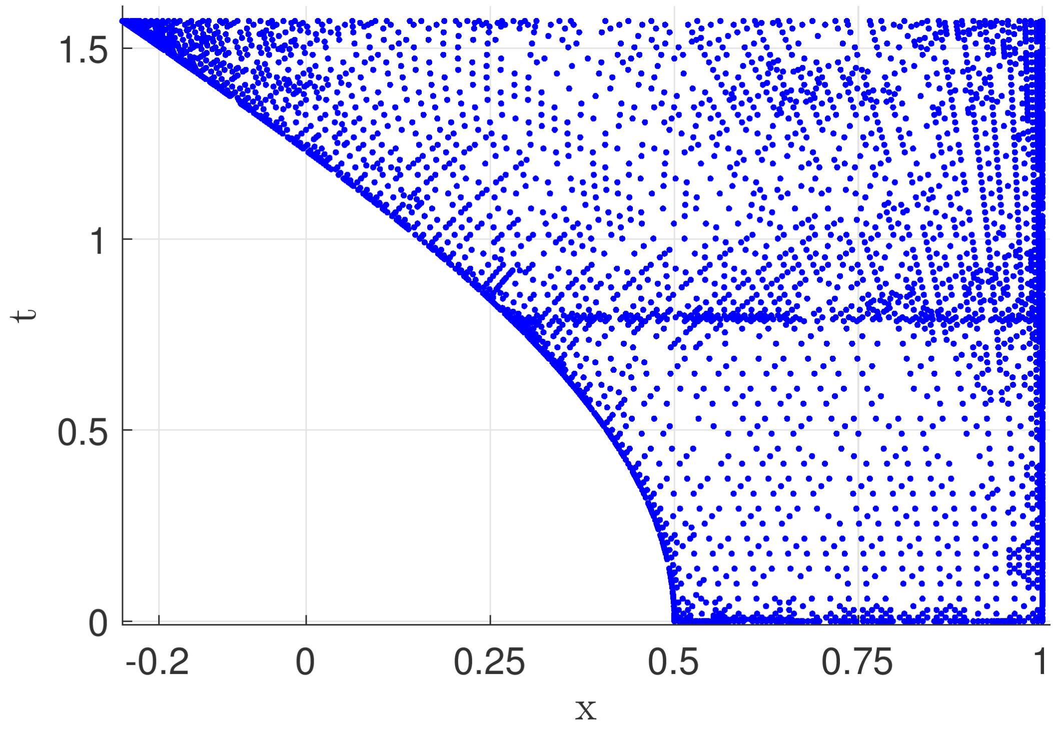
IC : $h(x,0) = f(x)$

BC : $h(1,t) = h(X(t),t) = h_0$

$q(1,t) = g_1(t)$

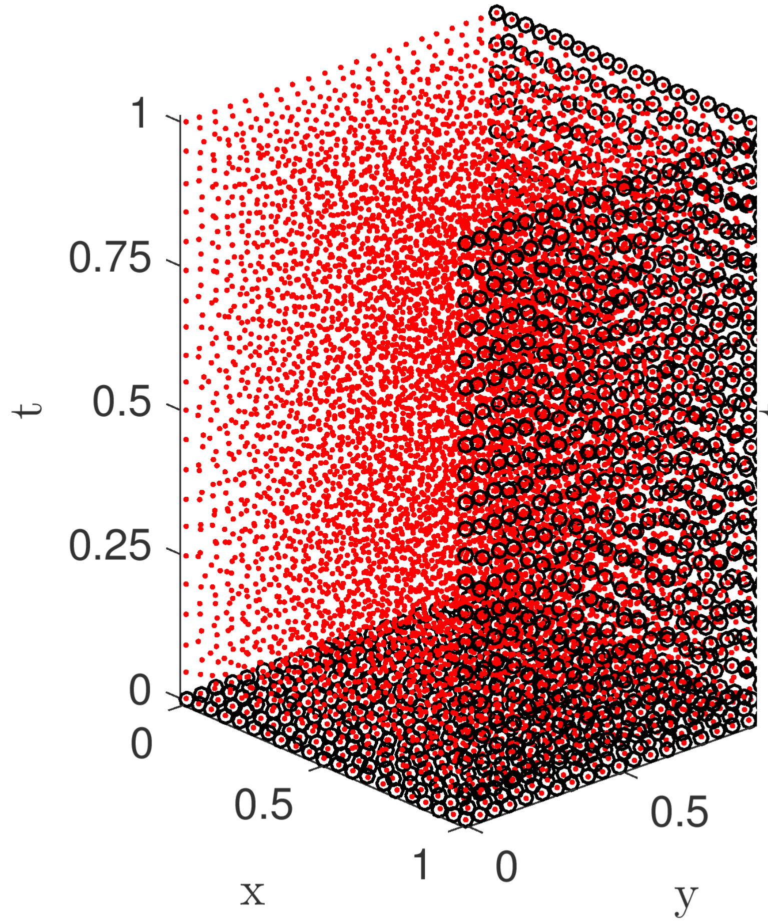
$q(X(t),t) = g_2(X(t),t)$

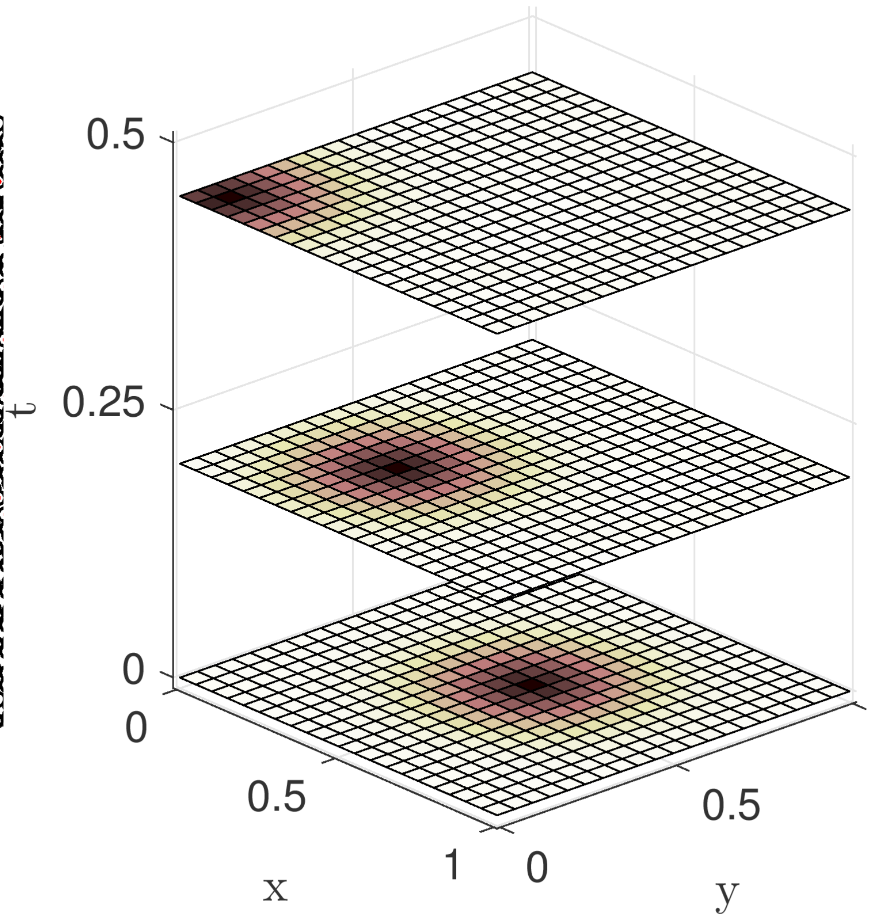


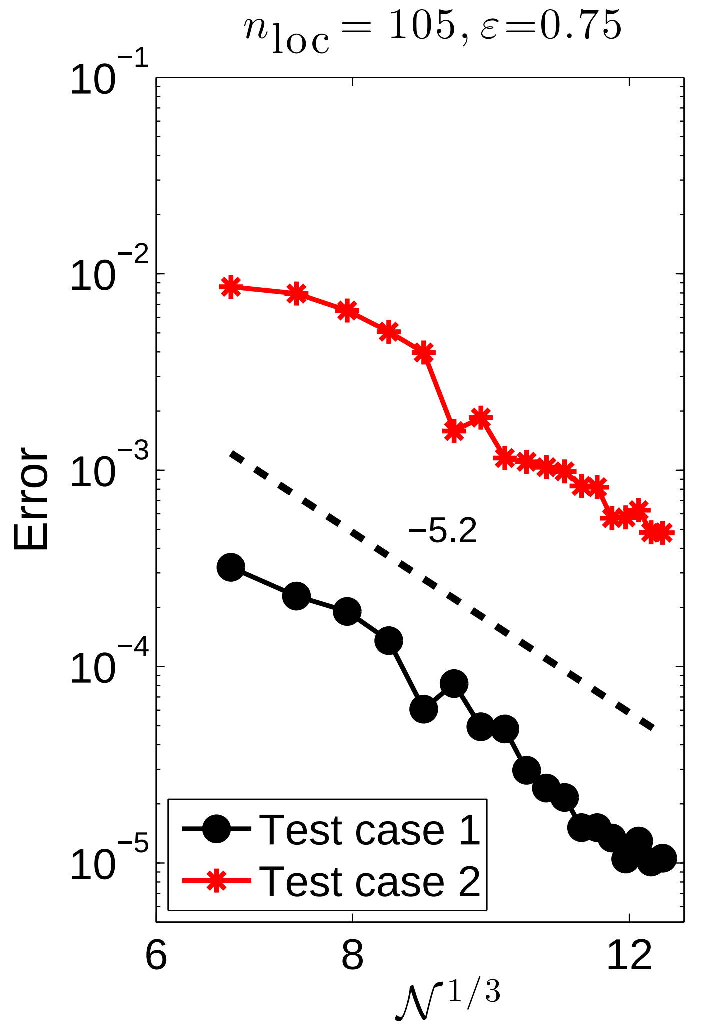


t+2D Advection Example

$$u_t = 0.5u_x + 0.75u_y + F(x,y,t) \quad (x,y) \in [0,1) \times [0,1)$$
$$u(1,y,t) = f_1(1,y,t) \quad u(x,1,t) = f_2(x,1,t)$$
$$u(x,y,0) = g(x,y)$$







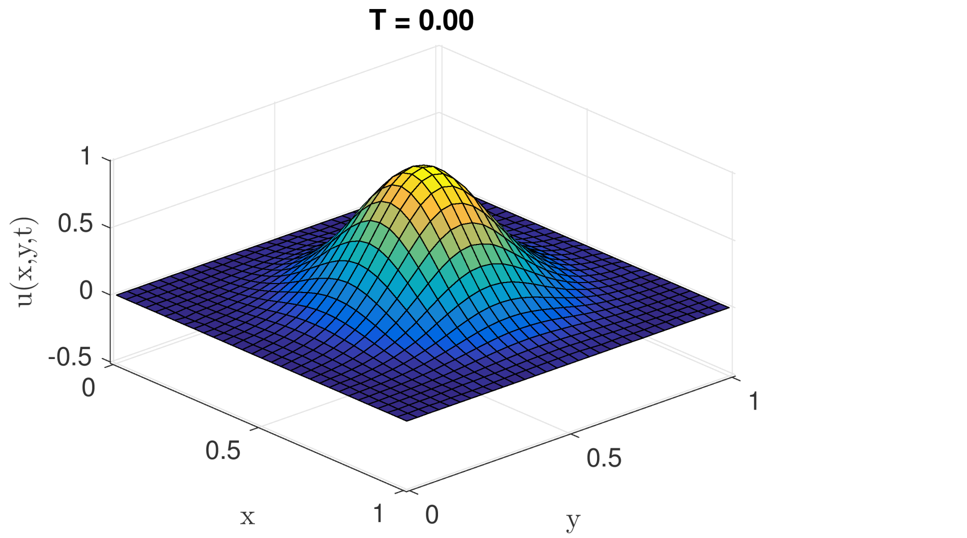
t+2D Wave Example

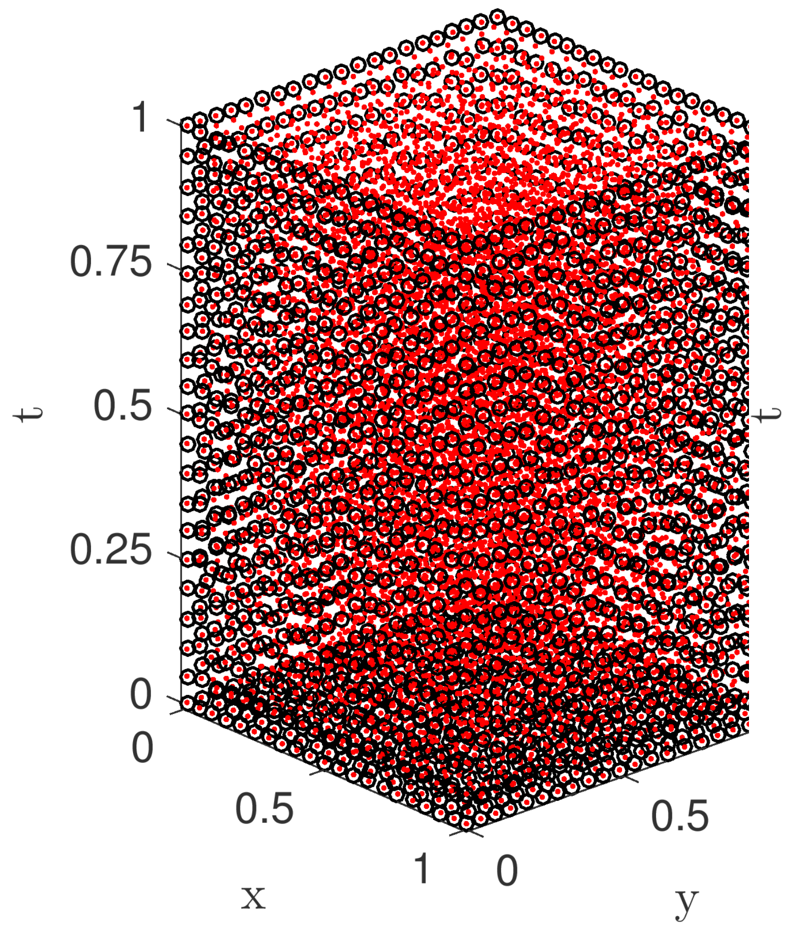
$$u_{tt} = \Delta u \quad (x, y) \in (0,1) \times (0,1)$$

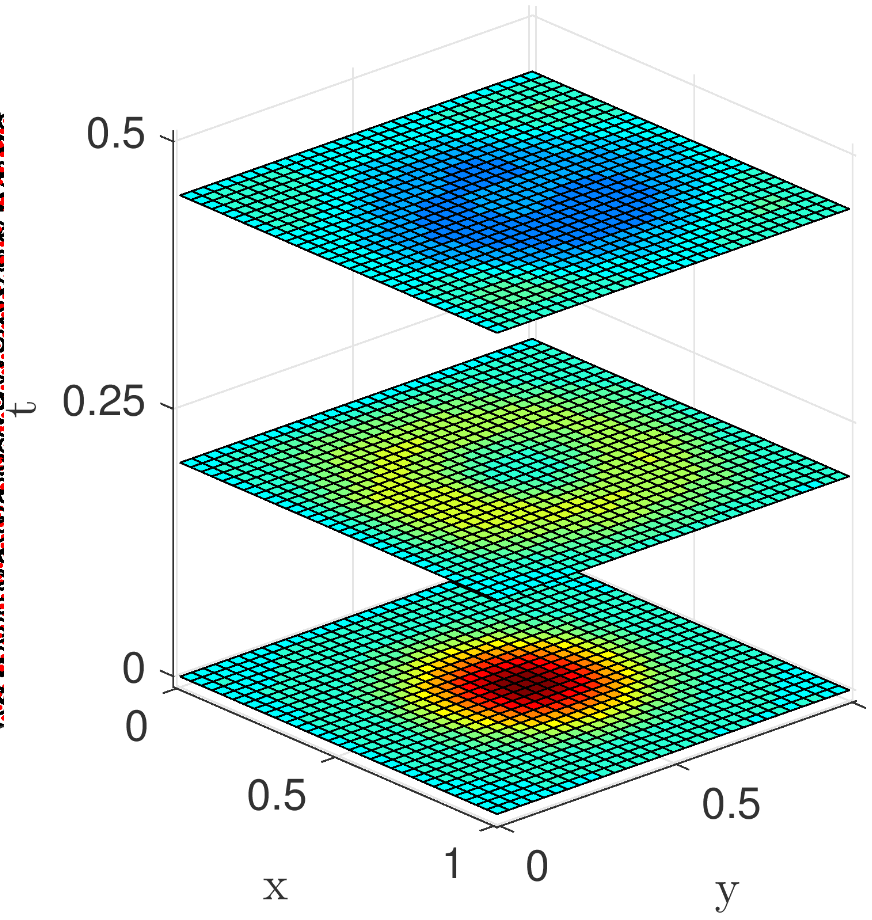
$$u(x,y,t) = 0 \quad \text{at the boundary}$$

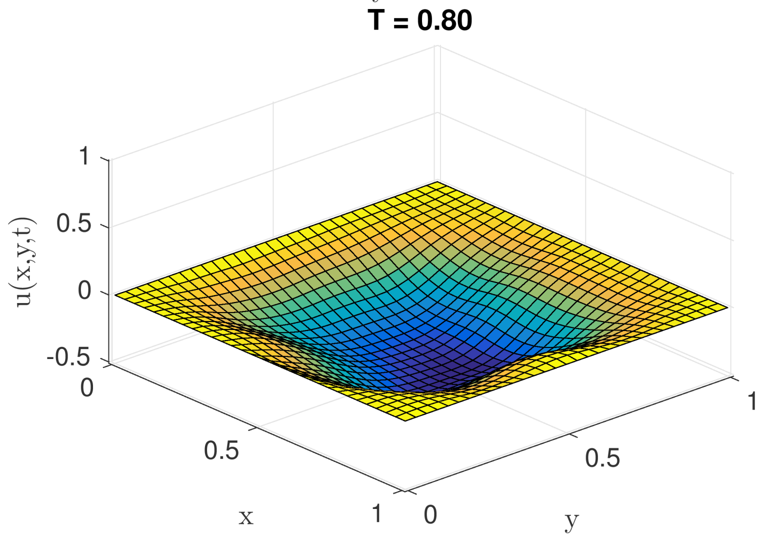
$$u(x,y,0) = g(x,y)$$

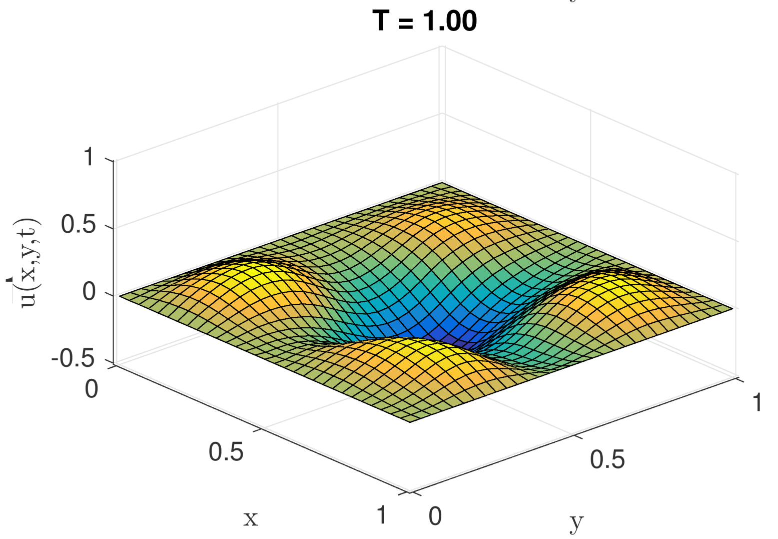
$$u_t(x,y,0) = 0$$







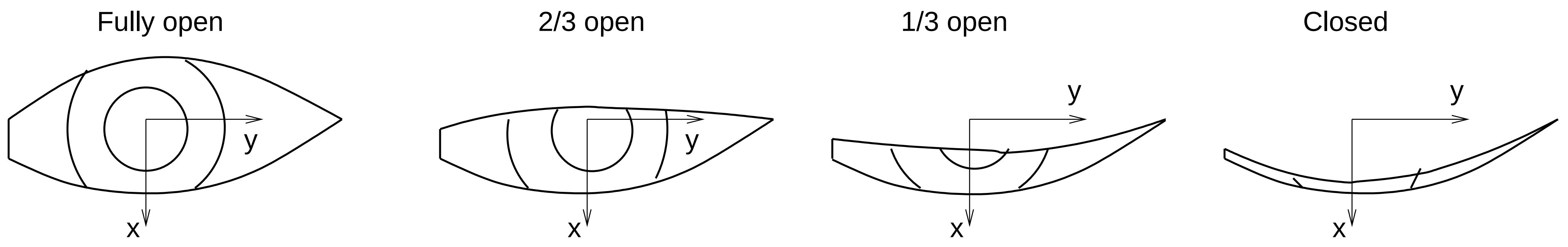




extra ghost/fictitious points for enforcing u_t

On-going study or future questions

- Stability: Can it only be done through adaptivity ?
- Least-Squares Space-time RBF-PU might be worth to try.
- Adaptivity in terms of partitions. Move away from points adaptivity.
- Preconditioner ?
- Possible GR application.
- Application to $2D + t$ Human Tear Film Dynamics.
- Enforce my grad students to finish the papers.


Fully open
2/3 open
1/3 open
Closed
x
y
x
y
x
y
x
y